INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 41, Northern Autumn 2019 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Let $P(x, y)$ be a polynomial such that for any integer $n \geq 0$ each of the polynomials $P(n, y)$ and $P(x, n)$ is either equal to the constant zero or has degree not greater than n. Is it possible that the polynomial $P(x, x)$ has an odd degree? (5 points)

2. Let ABC be an acute triangle. Suppose the points A', B', C' lie on the sides BC, AC, AB respectively and the line segments AA', BB', CC' intersect at a common point P inside the triangle. Three circles are constructed with diameters AA', BB' and CC' respectively and a chord through point P perpendicular to the corresponding diameter is constructed in each of the three circles. It is known that all three chords are of the same length. Prove that P is the intersection of the altitudes of triangle ABC. (5 points)

3. There are 100 visually identical coins of three types: gold, silver and bronze, with at least one coin of each type. Each gold coin weighs 3 grams, each silver coin weighs 2 grams and each bronze coin weighs 1 gram. How can one determine for sure the type of each coin by making no more than 101 weighings on a set of balance scales with no weights? (6 points)

4. A strictly increasing sequence of positive numbers that is infinite in both directions,

$$\cdots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \cdots,$$

is given. For a positive integer k, let b_k be the smallest integer such that the ratio of the sum of any k consecutive terms of the sequence to the largest of those k terms does not exceed b_k . Prove that the sequence $b_1, b_2, b_3, \ldots$ either coincides with the sequence of all positive integers $1, 2, 3, \ldots$ or is constant after some term. (10 points)

5. Let M be a point inside a convex quadrilateral $ABCD$ such that M is equidistant from the lines AB and CD and is also equidistant from the lines BC and AD. It is known that the area of $ABCD$ is equal to $MA \cdot MC + MB \cdot MD$. Prove that quadrilateral $ABCD$ is

(a) inscribed, i.e. cyclic. (6 points)

(b) circumscribed. (6 points)

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6. A cube consisting of $(2N)^3$ unit cubes is pierced by several needles parallel to the edges of the cube. Each needle pierces exactly $2N$ unit cubes and each unit cube is pierced by at least one needle.

(a) Prove that one can choose $2N^2$ needles going in at most two different directions such that no two of them pierce the same unit cube. (6 points)

(b) What is the largest number of needles that one can choose for sure such that no two of them pierce the same unit cube. (6 points)

7. Some of the numbers $1, 2, 3, \ldots, n$ have been painted red so that the following condition holds:

> if for red numbers a, b, c (not necessarily distinct), $a(b-c)$ is divisible by n, then $b = c$.

Prove there are no more than $\varphi(n)$ red numbers, where $\varphi(n)$ is the number of positive integers not exceeding n that are relatively prime to n. (12 points)